

ON DIRECTIONAL MAXIMAL OPERATORS IN HIGHER DIMENSIONS

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ABSTRACT. We introduce a notion of (finite order) lacunarity in higher dimensions for which we can bound the associated directional maximal operators in $L^p(\mathbb{R}^n)$, with $p > 1$. In particular, we are able to treat the (almost disjoint) classes previously considered by Nagel–Stein–Wainger, Sjögren–Sjölin and Carbery. Closely related to this, we find a characterisation of the sets of directions which give rise to bounded maximal operators. The bounds enable Lebesgue type differentiation of integrals in $L^p_{\text{loc}}(\mathbb{R}^n)$, replacing balls by tubes which point in these directions.

INTRODUCTION

For $n \geq 2$ and a set of directions Ω in the unit sphere \mathbb{S}^{n-1} , the directional maximal operator M_Ω is defined, initially on Schwartz functions, by

$$M_\Omega f(x) = \sup_{\omega \in \Omega} \sup_{r > 0} \frac{1}{2r} \int_{-r}^r |f(x - t\omega)| dt.$$

If Ω consists of a single direction and $p > 1$, the $L^p(\mathbb{R}^n)$ –boundedness of M_Ω follows from the Hardy–Littlewood maximal theorem. When Ω is a finite set, a fundamental problem is to find optimal bounds for the operator norm of M_Ω as a function of the cardinality of Ω and p . Allowing Ω to be infinite, one can also ask for conditions on the directions which ensure boundedness. In two dimensions, with directions in \mathbb{S}^1 , the questions have been answered with remarkable accuracy (see [7, 19, 12] for the first question, [18, 10, 15, 16] for the second question, or [2, 3, 11] which address the two questions in a unified way), however much less is known in higher dimensions (see [20, 5, 14] for the first question and [15, 16, 6] for the second).

For $\sigma \in \Sigma \equiv \Sigma(n) = \{(j, k) : 1 \leq j < k \leq n\}$ we consider $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ that satisfy $0 < \theta_{\sigma,i+1} \leq \lambda_\sigma \theta_{\sigma,i}$ with lacunary constants $0 < \lambda_\sigma < 1$, and for an orthonormal basis (e_1, \dots, e_n) , we divide the directions into *segments*

$$\Omega_{\sigma,i} = \left\{ \omega \in \Omega : \theta_{\sigma,i+1} < \left| \frac{\omega_k}{\omega_j} \right| \leq \theta_{\sigma,i} \right\}.$$

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Writing $\Omega_{\sigma,\infty} = \Omega \cap (e_j^\perp \cup e_k^\perp)$ and $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$, we prove the following theorem which is sharp in the sense that the supremum over partitions must be taken over the whole of Σ , and the segments must ‘accumulate’ at the hyperplanes perpendicular to the orthonormal basis vectors.

Theorem A. *Let $n \geq 2$ and $p > 1$. Then*

$$\|M_\Omega\|_{p \rightarrow p} \leq C \sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}^*} \|M_{\Omega_{\sigma,i}}\|_{p \rightarrow p},$$

where C depends only on n , p and the lacunary constants λ_σ for $\sigma \in \Sigma$.

As with the almost orthogonality principle of Alfonseca in two dimensions [1], we recover the best known results for the second question in higher dimensions. Nagel, Stein and Wainger [15] proved the L^p -boundedness of the maximal operator associated to the directions $\{(\vartheta_i^{a_1}, \dots, \vartheta_i^{a_n})\}_{i \geq 1}$ where $0 < a_1 < \dots < a_n$ and $0 < \vartheta_{i+1} \leq \lambda \vartheta_i$ with lacunary constant $0 < \lambda < 1$. We can apply Theorem A with $\theta_{\sigma,i} = \vartheta_i^{a_k - a_j}$, where $\sigma = (j, k)$, reducing the problem to that of a single direction. Note that it makes no difference if the directions are normalised to live on the unit sphere or not. On the other hand, Carbery [6] proved that the maximal operator associated to the directions $\{(2^{k_1}, \dots, 2^{k_n})\}_{k_1, \dots, k_n \in \mathbb{Z}}$ is L^p -bounded with $p > 1$. Taking $\theta_{\sigma,i} = 2^{-i}$, the resulting sets of directions $\Omega_{\sigma,i}$ are restricted to $(n-1)$ -dimensional hyperplanes, so that by choosing a suitable basis and applying Fubini’s theorem, we reduce to the $(n-1)$ -dimensional problem. Iterating the process we eventually end up with isolated directions as before.

In higher dimensions, it is not sufficient to constrain the angles between an infinite number of directions if they are to give rise to a bounded maximal operator. However Theorem A suggests a definition of lacunarity that gives rise to bounded maximal operators in general. An orthonormal basis of $\text{span}(\Omega) = \mathbb{R}^d$ and lacunary sequences $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ define partitions $\{\Omega_{\sigma,i}\}_{i \in \mathbb{Z}^*}$ for each $\sigma \in \Sigma(d)$. We call such a choice of $\frac{1}{2}d(d-1)$ partitions a *dissection*. Now if Ω consists of a single direction we call it *lacunary of order 0*. Recursively, we say that Ω is *lacunary of order L* if there is a dissection for which the segments $\Omega_{\sigma,i}$ are lacunary of order $\leq L-1$ for all $i \in \mathbb{Z}^*$ and $\sigma \in \Sigma(d)$, with uniformly bounded lacunary constants. According to this definition, the Nagel–Stein–Wainger directions are lacunary of order 1 and the Carbery directions are lacunary of order $n-1$. By repeatedly applying Theorem A as before, M_Ω is $L^p(\mathbb{R}^n)$ -bounded, with $p > 1$, whenever Ω is lacunary of finite order. This extends the two-dimensional result due to Sjögren–Sjölin [16] (the union of K sets of directions of lacunary order L with respect to their definition, is lacunary of order $2KL+1$ with respect to ours).

Bateman [4] proved that these are the only sets which give rise to bounded maximal operators in two dimensions. We do not know if this is true in higher dimensions, however we characterise the directions which give rise to bounded maximal operators using a formally larger class. We say that a

segment $\Omega_{\sigma, i_\sigma}$ of a partition $\{\Omega_{\sigma, i}\}_{i \in \mathbb{Z}^*}$ is *dominating* if it satisfies

$$\|M_{\Omega_{\sigma, i}}\|_{p \rightarrow p} \leq 2\|M_{\Omega_{\sigma, i_\sigma}}\|_{p \rightarrow p} \quad \text{for all } i \in \mathbb{Z}^*.$$

As before, if Ω consists of a single direction we call it *p-lacunary of order 0*. Recursively, we say that Ω is *p-lacunary of order L* if there is a dissection with a dominating segment for each of the partitions which is *p-lacunary of order $\leq L - 1$* . Again by Theorem A, M_Ω is $L^p(\mathbb{R}^n)$ -bounded, with $p > 1$, whenever Ω is *p-lacunary of finite order*. Note that it is not clear from the definition that any segment of a finite order *p-lacunary set* is *p-lacunary of finite order*, however this becomes clear in light of the following equivalence. We see that the directions which give rise to bounded maximal operators can be no worse than directions that can be divided into isolated directions by a finite number of lacunary dissections.

Theorem B. *Let $n \geq 2$ and $1 < p < \infty$. Then the following are equivalent:*

- (I) $\Omega \subset \mathbb{S}^{n-1}$ is *p-lacunary of finite order*
- (II) M_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Given an m -dimensional subspace $\Pi \subset \mathbb{R}^n$ and a set of directions Ω , we define the *shadow* Ξ of Ω on Π by

$$\Xi = \left\{ \frac{P_\Pi(\omega)}{|P_\Pi(\omega)|} : \omega \in \Omega \setminus \Pi^\perp \right\} \subset \Pi \cap \mathbb{S}^{n-1},$$

where P_Π denotes the orthogonal projection onto Π . Let $\text{Lac2shad}(n)$ denote the class of sets of directions whose 2-shadows are lacunary of finite order, where both the lacunary orders and lacunary constants are uniformly bounded. Similarly, we denote by $\text{Lac}(n)$ and $p\text{-Lac}(n)$ the classes of sets of directions which are lacunary and *p-lacunary of finite order*, respectively. The proof of Theorem B also yields the following inclusions.

Proposition C. *Let $n \geq 2$ and $1 < p < \infty$. Then*

$$\text{Lac}(n) \subset p\text{-Lac}(n) \subset \text{Lac2shad}(n).$$

With $n = 2$, these classes coincide of course, and so the sets of directions which give rise to $L^p(\mathbb{R}^2)$ -bounded maximal operators are the same for all $1 < p < \infty$. It is tempting to suppose that this is true in higher dimensions, however it may also be that $p\text{-Lac}(n)$ grows with p . In any case, we see that $p\text{-Lac}(n)$ is not so far from a purely two-dimensional concept. This should be compared with [13], where *Keakeya sets* in \mathbb{R}^3 with near minimal dimension were shown to have a ‘planiness’ property.

After a suitably fine finite splitting of the directions, the operator M_Ω can be composed with one-dimensional Hardy–Littlewood maximal operators to dominate a constant multiple of the maximal operator \mathcal{M}_Ω defined by

$$\mathcal{M}_\Omega f(x) = \sup_{x \in T \in \mathcal{T}_\Omega} \frac{1}{|T|} \int_T |f(y)| dy,$$

which in turn pointwise dominates M_Ω . Here, \mathcal{T}_Ω denotes the family of tubes which point in a direction of Ω . Standard density arguments yield a Lebesgue type differentiation result: If Ω is p -lacunary of finite order, then

$$\lim_{\substack{x \in T \in \mathcal{T}_\Omega \\ \text{diam}(T) \rightarrow 0}} \frac{1}{|T|} \int_T f(y) dy = f(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

for all $f \in L^p_{\text{loc}}(\mathbb{R}^n)$. For $1 < p < 2$, this convergence is in fact equivalent to weak type (p, p) bounds for \mathcal{M}_Ω by the Stein maximal principle [17]. Finally we note that by fixing the eccentricity (length/width) of the tubes \mathcal{T}_Ω and considering the associated maximal operator, directions Ω which give rise to unbounded M_Ω and \mathcal{M}_Ω can be considered. Indeed, Córdoba [8] proved that such a maximal operator is bounded, with a logarithmic dependency on the eccentricity, if the directions are restricted to a curve which intersects the hyperplanes of \mathbb{R}^n no more than a uniformly bounded number of times.

In the following section, we prove Theorem A which yields the implication (I) \Rightarrow (II) of Theorem B. The key ingredient is a somewhat nonlinear partition of a hyperplane in which we compensate for the points which are covered more than once by removing smaller sets. By adding and subtracting enough times we are able to partition the hyperplane with intersections of tensor products of two-dimensional cones. In the second section we prove Proposition C and that (II) \Rightarrow (I), which completes the proof of Theorem B. Unusually in this context, this follows by a topological argument. In the final section, we justify a number of remarks from above by constructing sets of apparently well-behaved directions for which the associated maximal operators are unbounded.

1. PROOF OF THEOREM A

By a finite splitting we can suppose in this section that the directions Ω are contained in the first open ‘octant’ of the unit sphere $\mathbb{S}^{n-1} \cap \mathbb{R}_+^n$. Note that the desired inequality is stronger if the segments $\Omega_{\sigma,i}$ are divided further, so by adding elements to our lacunary sequences if necessary, we can assume that $\frac{2}{3}\theta_{\sigma,i} \leq \theta_{\sigma,i+1}$ for all $i \in \mathbb{Z}$ and $\sigma \in \Sigma$. We then consider intersections of the segments to obtain cells of directions $\Omega_{\mathbf{i}} = \bigcap_{\sigma \in \Sigma} \Omega_{\sigma, i_\sigma}$ for each $\mathbf{i} = (i_\sigma)_{\sigma \in \Sigma} \in \mathbb{Z}^\Sigma$. This yields a finer partition than those of the introduction;

$$\Omega = \bigcup_{\mathbf{i} \in \mathbb{Z}^\Sigma} \Omega_{\mathbf{i}} \quad \text{so that} \quad M_\Omega = \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} M_{\Omega_{\mathbf{i}}}.$$

Note that many of the cells are empty, however we will see that this over-determination is somehow unavoidable. Let $K_{\sigma,i}$ denote the convolution operator associated to a smooth Fourier multiplier $\psi_{\sigma,i}$, equal to one on

$$\Psi_{\sigma,i} = \left\{ \xi \in \mathbb{R}^n : |\theta_{\sigma,i}\xi_k + \xi_j| \leq \frac{n-1}{n} (|\theta_{\sigma,i}\xi_k| + |\xi_j|) \right\},$$

and supported in a similar cone with the constant $\frac{n-1}{n}$ replaced by $\frac{n}{n+1}$. Note that the aperture narrows as $|i|$ grows.

The key geometric fact used in the proof of the following lemma is that the hyperplane perpendicular to ω is contained in $\cup_{\sigma \in \Sigma} \Psi_{\sigma, i_\sigma}$ for all $\omega \in \Omega_i$. When $n \geq 3$, this fails to be true if we replace $\frac{n-1}{n}$ in the definition of $\Psi_{\sigma, i}$ by another constant strictly less than $\frac{n-2}{n}$, no matter how fine we take the partition. The proof of Theorem A hinges upon the fact that this covering is possible with a constant strictly less than one. At this point, the lacunarity is not used, just that the partition is sufficiently fine.

Lemma 1.1. *Let $p > 1$. Then there is a C , depending only on n, p , so that*

$$\|M_\Omega\|_{p \rightarrow p} \leq C \sup_{\emptyset \neq \Gamma \subset \Sigma} \left\| \sup_{i \in \mathbb{Z}^\Sigma} M_{\Omega_i} \prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right\|_{p \rightarrow p}.$$

Proof. Fix a nonnegative even Schwartz function m_o^\vee which is positive on $[-1, 1]$, so that, for positive functions, $M_\Omega f$ is pointwise equivalent to

$$\sup_{\omega \in \Omega} \sup_{r > 0} \left| \frac{1}{r} \int m_o^\vee\left(\frac{t}{r}\right) f(\cdot - t\omega) dt \right|.$$

Throughout, $^\wedge$ and $^\vee$ denote the Fourier transform and inverse transform, respectively. One can calculate that

$$\left(\frac{1}{r} \int m_o^\vee\left(\frac{t}{r}\right) f(\cdot - t\omega) dt \right)^\wedge(\xi) = m_o(r\omega \cdot \xi) f^\wedge(\xi).$$

It will simplify things to take m_o supported in $[-1, 1]$, which can be arranged by choosing $m_o = \phi_o * \phi_o$ where ϕ_o is an even Schwartz function supported in $[-1/2, 1/2]$. We also fix a Schwartz function η_o , supported in the ball of radius $4n^2$, centred at the origin and equal to one on the concentric ball of radius $2n^2$, and consider the operator

$$f \mapsto \sup_{\omega \in \Omega} \sup_{r > 0} |S_{r, \omega} f|,$$

where $(S_{r, \omega} f)^\wedge(\xi) = \eta_o(r(\omega_1 \xi_1, \dots, \omega_n \xi_n)) m_o(r\omega \cdot \xi) f^\wedge(\xi)$. This is pointwise dominated by a constant multiple of the strong maximal operator \mathcal{M}_{str} , which can be bounded by iterated applications of the one-dimensional Hardy–Littlewood maximal theorem. Defining m by $m(\xi) = (1 - \eta_o)(\xi) m_o(\mathbf{1} \cdot \xi)$ with $\mathbf{1} = (1, \dots, 1)$, we are left with the maximal operator T_Ω defined by

$$f \mapsto \sup_{\omega \in \Omega} \sup_{r > 0} |T_{r, \omega} f|,$$

where $(T_{r, \omega} f)^\wedge(\xi) = m(r(\omega_1 \xi_1, \dots, \omega_n \xi_n)) f^\wedge(\xi)$. A variant of this decomposition was originally employed by Nagel, Stein and Wainger [15].

It will suffice to prove the pointwise estimate

$$(1) \quad T_\Omega f \leq \sum_{\emptyset \neq \Gamma \subset \Sigma} \sup_{i \in \mathbb{Z}^\Sigma} T_{\Omega_i} \left[\prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right] f.$$

The desired L^p -estimate then follows by combining with the inequalities

$$M_\Omega f \leq C(\mathcal{M}_{\text{str}} f + T_\Omega f), \quad T_{\Omega_i} f \leq C(\mathcal{M}_{\text{str}} f + M_{\Omega_i} f),$$

and, when $\Omega_i \neq \emptyset$,

$$\left\| \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} \mathcal{M}_{\text{str}} \prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right\|_{p \rightarrow p} \leq C \left\| \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} M_{\Omega_i} \prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right\|_{p \rightarrow p}.$$

The final inequality is a trivial consequence of the boundedness of the strong maximal operator, combined with the fact that $|f| \leq M_{\Omega_i} f$.

In order to prove (1), we consider the multiplier operators \mathcal{R}_i defined by

$$(\mathcal{R}_i f)^\wedge(\xi) = \prod_{\sigma \in \Sigma} (1 - \psi_{\sigma, i_\sigma})(\xi) f^\wedge(\xi) \quad \text{for all } \mathbf{i} = (i_\sigma)_{\sigma \in \Sigma}.$$

Expanding $1 - \prod_{\sigma} (1 - x_\sigma)$, we see that

$$T_{r, \omega} f = T_{r, \omega} \mathcal{R}_i f + \sum_{\emptyset \neq \Gamma \subset \Sigma} (-1)^{|\Gamma|+1} T_{r, \omega} \left[\prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right] f.$$

In contrast with the operators $K_{\sigma, i}$, which are essentially two-dimensional, the operators \mathcal{R}_i are genuinely higher dimensional objects, however we will prove that the multiplier associated to $T_{r, \omega} \mathcal{R}_i$ is identically zero whenever $\omega \in \Omega_i$ and $r > 0$;

$$(2) \quad m(r(\omega_1 \xi_1, \dots, \omega_n \xi_n)) \prod_{\sigma \in \Sigma} (1 - \psi_{\sigma, i_\sigma})(\xi) \equiv 0.$$

Recalling that $T_\Omega f = \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} T_{\Omega_i} f$, this yields (1).

In order to prove (2) whenever $\omega \in \Omega_i$ and $r > 0$, after the scaling $\omega_j \xi_j \rightarrow \xi_j$ for $1 \leq j \leq n$, it will suffice to prove that the region defined by

$$(3) \quad \left| \sum_{j=1}^n \xi_j \right| \leq \frac{1}{r} \quad \text{and} \quad \left(\sum_{j=1}^n \xi_j^2 \right)^{1/2} \geq \frac{2n^2}{r}$$

$$\left| \theta_{\sigma, i_\sigma} \frac{\xi_k}{\omega_k} + \frac{\xi_j}{\omega_j} \right| > \frac{n-1}{n} \left(\theta_{\sigma, i_\sigma} \left| \frac{\xi_k}{\omega_k} \right| + \left| \frac{\xi_j}{\omega_j} \right| \right) \quad \text{for all } \sigma \in \Sigma$$

is empty. The boundary of the cones are the hyperplanes defined by

$$\theta_{\sigma, i_\sigma} \frac{\xi_k}{\omega_k} = -(2n-1)^{\pm 1} \frac{\xi_j}{\omega_j}.$$

As $\frac{2}{3} \theta_{\sigma, i_\sigma} \leq \frac{\omega_k}{\omega_j} < \theta_{\sigma, i_\sigma}$, these cones are properly contained in the cones with boundary given by $\xi_k = -n^{\pm 1} \xi_j$. That is to say, they are contained in

$$(4) \quad |\xi_k + \xi_j| > \frac{n-1}{n+1} (|\xi_k| + |\xi_j|) \quad \text{for all } (j, k) \in \Sigma.$$

We suppose for a contradiction that the region defined by (3) and (4) is not empty. It is clear by comparing the inequalities in (3) that the components of a vector ξ in this region cannot all have the same sign. By symmetric invariance of the conditions, we may suppose that $\xi_1, \dots, \xi_{m-1} \geq 0$ and that $\xi_m, \dots, \xi_n < 0$ for some $1 < m \leq n$. We can also suppose without loss of

generality that $|\xi_1| \geq |\xi_j|$ for all $j > 1$ and $|\xi_m| \geq |\xi_j|$ for all $j > m$. Then taking $j = 1$ and $k = m$ in (4);

$$\xi_1 + \xi_m > \frac{n-1}{n+1}(\xi_1 - \xi_m),$$

which gives $|\xi_1| \geq n|\xi_m|$. On the other hand, by the first condition of (3),

$$|\xi_1| - (n-1)|\xi_m| \leq \left| \sum_{j=1}^n \xi_j \right| \leq \frac{1}{r}.$$

Combining the two estimates we obtain $|\xi_1| \leq n/r$. Since $|\xi_1| \geq |\xi_j|$ for $j > 1$, this yields $|\xi_1| + \dots + |\xi_n| \leq n^2/r$ which contradicts the second inequality in (3). Thus, $T_{r,\omega}\mathcal{R}_{\mathbf{i}} \equiv 0$ whenever $r > 0$ and $\omega \in \Omega_{\mathbf{i}}$, and we are done. \square

We will also require the following square function estimates which follow easily from the two-dimensional theory.

Lemma 1.2. *Let $1 < p < \infty$ and $\Gamma \subset \Sigma$. Then*

$$\left\| \left(\sum_{\mathbf{i} \in \mathbb{Z}^\Gamma} \left| \prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma} \right| f \right)^2 \right\|_p^{\frac{1}{2}} \leq C \|f\|_p, \quad \text{where } \mathbf{i} = (i_\sigma)_{\sigma \in \Gamma},$$

and C depends only on $|\Gamma|$, p and the lacunary constants λ_σ for $\sigma \in \Gamma$.

Proof. In order to bound directional maximal operators for $p \geq 2$, it suffices to prove Theorem A for $p = 2$, in which case the square function estimate follows directly from Plancherel's theorem and the finite overlapping of the cones $\{\Psi_{\sigma, i}\}_{i \in \mathbb{Z}}$. This is where we use the lacunarity of the sequences $\{\theta_{\sigma, i}\}_{i \in \mathbb{Z}}$. When $p \neq 2$, by a standard randomising argument, using Khintchine's inequality, the square function estimates follow from the uniform L^p -boundedness, independent of the choice of the signs, of the Fourier multiplier operators

$$f \mapsto \left(\sum_{\mathbf{i} \in \mathbb{Z}^\Gamma} \pm \prod_{\sigma \in \Gamma} \psi_{\sigma, i_\sigma} f^\wedge \right)^\vee.$$

This in turn is a consequence of the Marcinkiewicz multiplier theorem, for which it suffices to check a number of conditions involving integrals of derivatives of the multipliers. After applying the product rule, the calculation reduces to the case $|\Gamma| = 1$. Applying Fubini's theorem so as to ignore the trivial variables, this was originally checked by Córdoba and Fefferman [9, Section 4] in their proof of a two-dimensional angular Littlewood-Paley inequality (see also [15]). \square

Armed with these lemmas, the proof is completed easily as follows:

Case $p \geq 2$. We consider $\mathbb{Z}^\Sigma = \mathbb{Z}^\Gamma \times \mathbb{Z}^{\Sigma \setminus \Gamma}$, and given $\mathbf{i} = (i_\sigma)_{\sigma \in \Sigma}$, we write $\mathbf{i} = \mathbf{j} \times \mathbf{k}$ where $\mathbf{j} = (i_\sigma)_{\sigma \in \Gamma}$ and $\mathbf{k} = (i_\sigma)_{\sigma \in \Sigma \setminus \Gamma}$. Using the inclusion

$\ell^p(\mathbb{Z}^\Gamma) \hookrightarrow \ell^\infty(\mathbb{Z}^\Gamma)$ and interchanging the order of the sum and the integral,

$$\begin{aligned}
 (5) \quad \left\| \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_p &\leq \left(\sum_{\mathbf{j} \in \mathbb{Z}^\Gamma} \left\| \sup_{\mathbf{k} \in \mathbb{Z}^{\Sigma \setminus \Gamma}} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_p^p \right)^{\frac{1}{p}} \\
 &\leq \sup_{\mathbf{j} \in \mathbb{Z}^\Gamma} \left\| \sup_{\mathbf{k} \in \mathbb{Z}^{\Sigma \setminus \Gamma}} M_{\Omega_{\mathbf{i}}} \right\|_{p \rightarrow p} \left(\sum_{\mathbf{j} \in \mathbb{Z}^\Gamma} \|f_{\mathbf{j}}\|_p^p \right)^{\frac{1}{p}} \\
 &\leq \sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}} \|M_{\Omega_{\sigma, i}}\|_{p \rightarrow p} \left\| \left(\sum_{\mathbf{j} \in \mathbb{Z}^\Gamma} |f_{\mathbf{j}}|^p \right)^{\frac{1}{p}} \right\|_p.
 \end{aligned}$$

Taking $f_{\mathbf{j}} = [\prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma}] f$, where $\mathbf{j} = (i_\sigma)_{\sigma \in \Gamma}$, and applying Lemmas 1.1 and 1.2, we obtain the desired estimate.

Case $1 < p < 2$. This is based on an argument of Christ used in [6, 1] which refined the argument of Nagel–Stein–Wainger [15]. We suppose initially that Ω is finite, so that by the triangle inequality and the Hardy–Littlewood maximal theorem, M_Ω is bounded. Then interpolation between

$$\left\| \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_p \leq \|M_\Omega\|_{p \rightarrow p} \left\| \sup_{\mathbf{j} \in \mathbb{Z}^\Gamma} |f_{\mathbf{j}}| \right\|_p$$

and (5), yields

$$\left\| \sup_{\mathbf{i} \in \mathbb{Z}^\Sigma} M_{\Omega_{\mathbf{i}}} f_{\mathbf{j}} \right\|_p \leq \|M_\Omega\|_{p \rightarrow p}^{1-\frac{p}{2}} \left(\sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}} \|M_{\Omega_{\sigma, i}}\|_{p \rightarrow p} \right)^{\frac{p}{2}} \left\| \left(\sum_{\mathbf{j} \in \mathbb{Z}^\Gamma} |f_{\mathbf{j}}|^2 \right)^{\frac{1}{2}} \right\|_p.$$

Taking $f_{\mathbf{j}} = [\prod_{\sigma \in \Gamma} K_{\sigma, i_\sigma}] f$, where $\mathbf{j} = (i_\sigma)_{\sigma \in \Gamma}$, and applying Lemmas 1.1 and 1.2 as before, we see that

$$\|M_\Omega\|_{p \rightarrow p} \leq C \|M_\Omega\|_{p \rightarrow p}^{1-\frac{p}{2}} \left(\sup_{\sigma \in \Sigma} \sup_{i \in \mathbb{Z}} \|M_{\Omega_{\sigma, i}}\|_{p \rightarrow p} \right)^{\frac{p}{2}}.$$

Rearranging, we obtain the desired estimate with C independent of Ω , so we can drop the restriction that Ω is finite, which completes the proof. \square

In both [15] and [6], a single conic Fourier multiplier was introduced for each direction. This multiplier had to cover (the bulk of) the hyperplane perpendicular to the direction, and so was necessarily multidimensional in nature. Restrictions on the directions then ensured finite overlapping of the supports of the multipliers, yielding a bound via orthogonality as above. Instead, we introduced a number of essentially two-dimensional multipliers which added flexibility at the same time as simplifying matters. It is necessary to add and subtract a number of products of these multipliers in order to obtain a partition (a cover being inadequate), however this came at essentially no cost. This simplifies matters because the orthogonality in two dimensions is much easier to see (for example it is checked while summing over only one index). On the other hand, the multipliers are in some sense

stable under perturbation allowing us to introduce one for each segment instead of one for each direction.

2. PROOF OF THEOREM B AND PROPOSITION C

Theorem A enables us to conclude that $p\text{-Lac}(n)$ is contained in the class of sets of directions which give rise to bounded maximal operators. Thus it remains to prove that (II) \Rightarrow (I) and $p\text{-Lac}(n) \subset \text{Lac2shad}(n)$. The latter is a direct consequence of the following lemma in which we appeal to a quantitative version of Bateman's theorem [4], allowing us to treat the 2-shadows simultaneously and thus uniformly. We also use that the cross product of a two-dimensional Kakeya set with a cube is a Kakeya set.

Lemma 2.1. *Let $n \geq 2$ and $1 < p < \infty$, and suppose that M_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Then $\Omega \in \text{Lac2shad}(n)$.*

Proof. As M_Ω is bounded if and only if $M_{\overline{\Omega}}$ is bounded, we can suppose that Ω is closed. We appeal to Bateman's terminology [4]. In particular we will consider the binary tree \mathcal{T}_Π associated to the 2-shadow of Ω on Π , for any two-dimensional subspace Π , and their splitting numbers $\text{split}(\mathcal{T}_\Pi)$. We say that Ω admits Kakeya 2-shadows if there exists a constant C such that for any $N \geq 1$ there exists a two-dimensional subspace $\Pi(N)$ and a finite collection of rectangles $\mathcal{R}_{\Pi(N)}$ contained in $\Pi(N)$, with longest side pointing in a direction of the shadow of Ω on $\Pi(N)$, that satisfy

$$(6) \quad \left| \bigcup_{R \in \mathcal{R}_{\Pi(N)}} R \right| \leq \frac{C}{N} \left| \bigcup_{R \in \mathcal{R}_{\Pi(N)}} 3R \right|.$$

Here, $3R$ has the same center and width as R , but three times the length.

We prove the contrapositive. If $\Omega \notin \text{Lac2shad}(n)$, then by Theorem 3 (combined with Remark 2) in [4], for any $N \geq 1$, there is a shadow of Ω on $\Pi(N)$ for which $\text{split}(\mathcal{T}_{\Pi(N)}) \geq 2^N$. Bateman proved (see pages 61–62 and Claim 7 of [4]) that $\text{split}(\mathcal{T}_{\Pi(N)}) \geq 2^N$ implies the existence of a finite family $\mathcal{R}_{\Pi(N)}$ of rectangles R satisfying (6). Now for each $N \in \mathbb{N}$, we pick an orthonormal basis (e_1, \dots, e_n) so that $\text{span}(e_1, e_2) = \Pi(N)$. For each rectangle R in the subcollection $\mathcal{R}_{\Pi(N)}$, we set

$$\beta \equiv \beta(R) = \text{diam}(R)(\omega_1^2 + \omega_2^2)^{-1/2},$$

where ω is a direction of Ω whose 2-shadow points in the direction of R , and let $\alpha \equiv \alpha(N)$ to be ten times the maximum $\beta(R)$ with $R \in \mathcal{R}_{\Pi(N)}$. Taking

$$E_N = \bigcup_{R \in \mathcal{R}_{\Pi(N)}} R \times [0, \alpha]^{n-2},$$

defined with respect to the basis (e_1, \dots, e_n) , we then have

$$M_\Omega[\chi_{E_N}](x) \geq 1/8 \quad \text{for all } x \in \bigcup_{R \in \mathcal{R}_{\Pi(N)}} 3R \times [3\beta, \alpha - 3\beta]^{n-2}.$$

Using (6), we see that for all $N \geq 1$,

$$\|M_\Omega[\chi_{E_N}]\|_p \geq cN^{\frac{1}{p}} \|\chi_{E_N}\|_p,$$

so that M_Ω is not bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when p is finite. \square

Naively, this raises the hope that Theorem A can be applied repeatedly in order to reduce a 2-shadow to a single direction, thus reducing the dimension of the problem. However the lacunary orders of the shadows are unstable in the sense that shadows on two-dimensional subspaces which are close can have dramatically different lacunary orders, and so it is not clear that it helps to apply Theorem A and then change the basis in order to apply it again. One may be faced each time with lacunary orders which are as bad as before, and so the process may never end.

We side-step this problem using a topological argument. For an m -shadow Ξ , we consider $A_\ell = \text{Ac}(A_{\ell-1})$, where $A_1 = \text{Ac}(\Xi)$, the accumulation points of Ξ . We say that Ξ has *finite accumulation order* if there exists a finite $L \geq 1$ such that A_L has finite cardinality.

Lemma 2.2. *Let $2 \leq m \leq n$ and $1 < p < \infty$, and suppose that M_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Then the m -shadows of Ω have finite accumulation order.*

Proof. As M_Ω is bounded, the 2-shadows are uniformly lacunary of finite order by Lemma 2.1, so that in particular the 2-shadows of Ω have uniformly bounded accumulation order. Thus, it will suffice to prove that if the accumulation order of an m -shadow Ξ of Ω on Π is greater than L , then there exists a 2-shadow of Ξ , and hence also of Ω , whose accumulation order is greater than L . We take $\xi \in A_{L+1}$ and consider a sequence $\{\xi_j\}_{j \geq 1}$ in A_L which accumulates at ξ . Then for all but (at most) one $(m-1)$ -dimensional subspace of Π , the shadows of $\{\xi_j\}_{j \geq 1}$ on the $(m-1)$ -dimensional subspaces accumulate at the shadows of ξ . Then we consider sequences in A_{L-1} which accumulate at ξ_j . Again for all but one $(m-1)$ -dimensional subspace of Π , the shadows on the $(m-1)$ -dimensional subspaces accumulate at the shadows of ξ_j . Continuing the process, we see that for all but a countable number of $(m-1)$ -dimensional subspaces of Π , the shadow of ξ is of accumulation order $\geq L+1$. We take one such shadow and repeat the process. This yields an $(m-2)$ -dimensional shadow of the $(m-1)$ -dimensional shadow of ξ , which is an $(m-2)$ -dimensional shadow of ξ , that is of accumulation order $\geq L+1$. Repeating the process, we obtain the desired result. \square

We say that an m -shadow Ξ of Ω is (n, p) -lacunary of order 0 if Ξ consists of a single direction. Recursively, we say that Ξ is (n, p) -lacunary of order L if there are members $\{\Xi_{\sigma, i_\sigma}\}_{\sigma \in \Sigma(d)}$ of a dissection of Ξ which are (n, p) -lacunary of order $\leq L-1$ and for which the segments $\{\Omega_{\sigma, i_\sigma}\}_{\sigma \in \Sigma(d)}$ that shade them are dominating;

$$\|M_{\Omega_{\sigma, i}}\|_{p \rightarrow p} \leq 2 \|M_{\Omega_{\sigma, i_\sigma}}\|_{p \rightarrow p} \quad \text{for all } i \in \mathbb{Z}^*.$$

If an m -shadow Ξ of Ω is (n, p) -lacunary of finite order we will simply say that Ξ is (n, p) -lacunary. Now when M_Ω is bounded, the existence of dominating segments is always given, and so the fact that the 2-shadows are uniformly lacunary of order $\leq L$, implies that they are also (n, p) -lacunary of order $\leq L$. Thus, the implication (II) \Rightarrow (I) is obtained by combining Lemma 2.1 with $n - 2$ applications of the following lemma, observing that (n, p) -lacunary n -shadows are p -lacunary.

Lemma 2.3. *Let $2 \leq m \leq n - 1$ and $1 < p < \infty$, and suppose that M_Ω is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Then, if the m -shadows of Ω are (n, p) -lacunary, then the $(m + 1)$ -shadows of Ω are (n, p) -lacunary.*

Proof. As M_Ω is bounded, the $(m + 1)$ -shadows of Ω have finite accumulation order by Lemma 2.2. We suppose for a contradiction that the m -shadows of Ω are (n, p) -lacunary, but that there is an $(m + 1)$ -shadow of Ω which is not. Thus, we will have our desired contradiction if we can show that this $(m + 1)$ -shadow of Ω , which from now on we call Ξ , is of arbitrarily large accumulation order.

We take a basis (e_1, \dots, e_{m+1}) with e_{m+1} being an accumulation point of Ξ (which exists by compactness as finite sets are (n, p) -lacunary when M_Ω is bounded) and write $\Pi = \text{span}(e_1, \dots, e_m)$. Note that the m -shadow of Ξ on Π is the same as the m -shadow of Ω on Π . In fact we choose the basis vectors (e_1, \dots, e_m) more carefully: We dissect Ξ (simultaneously dissecting the m -shadow on Π and partially dissecting Ω) with (e_1, \dots, e_m) and $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ for each $\sigma \in \Sigma(m)$ chosen in order to reduce the (n, p) -lacunary order of the m -shadow on Π (we are free to choose any lacunary $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ for $\sigma = (j, m + 1)$ with $1 \leq j \leq m$). Now as the $(m + 1)$ -shadow is not (n, p) -lacunary, there must be a $\sigma_1 \in \Sigma(m + 1)$ for which the $(m + 1)$ -shadows on $\text{span}(e_1, \dots, e_{m+1})$ of the dominating segments $\Omega_{\sigma_1, i_{\sigma_1}}$ are not (n, p) -lacunary. Note that there are dominating segments in each partition as M_Ω is bounded. If $\sigma_1 = (j, m + 1)$ we have found such a shadow which is separated from e_{m+1} , and hence we have found a new accumulation point by compactness. If not, we choose one of the dominating segments whose m -shadow on Π has reduced (n, p) -lacunary order. Supposing that the span of this m -shadow is d_2 -dimensional, where $d_2 \leq d_1 \equiv m$, we take e_{d_2+1} to be the original accumulation point and dissect the $(m + 1)$ -shadow with $e_1, \dots, e_{d_2} \in \Pi$ and $\{\theta_{\sigma,i}\}_{i \in \mathbb{Z}}$ for $\sigma \in \Sigma(d_2)$ chosen in order to reduce the (n, p) -lacunary order of the m -shadow on Π of this segment. Again there is a $\sigma_2 \in \Sigma(d_2 + 1)$ for which the $(m + 1)$ -shadows of the dominating segments are not (n, p) -lacunary. If $\sigma_2 = (j, d_2 + 1)$ we have separated from the accumulation point. If not, we choose one of the dominating segments whose m -shadow on Π has reduced (n, p) -lacunary order and continue. This division into ever smaller sets of directions, whose $(m + 1)$ -shadow is not (n, p) -lacunary, cannot stop, as otherwise Ξ would be (n, p) -lacunary of finite order. Also, the segments which are not (n, p) -lacunary cannot be sliced using hyperplanes which pass through the accumulation point indefinitely as the (n, p) -lacunary order of

the m -shadow on Π is reduced at each slice so eventually we would reduce to the case where the m -shadow would be a single direction. In that case the $(m+1)$ -shadow would be contained in a two-dimensional subspace, and so would have to be (n, p) -lacunary by Lemma 2.1. We are choosing the segments to be of infinite (n, p) -lacunary order, and so the only way we can keep doing this is if one is eventually chosen which is disconnected from the accumulation point. As this segment has an infinite number of directions, by compactness we find a new accumulation point.

For a finite number of accumulation points, we can always make a judicious choice of basis and lacunary sequences so that they are separated in the dissection. Either this yields a segment whose $(m+1)$ -shadow is not (n, p) -lacunary which contains a new accumulation point, or there is a segment whose $(m+1)$ -shadow is not (n, p) -lacunary which contains one of the old accumulation points. In this case, we take e_{m+1} to be this contained accumulation point and repeat the process. At some stage the repetition of the process yields a segment which is not (n, p) -lacunary and which is disconnected from the contained accumulation point (and the others). This follows again by hypothesis because the segments produced in the dissection which are not (n, p) -lacunary can only contain the contained accumulation point a finite number of times. Continuing the process, we obtain a sequence of accumulation points, which by compactness have an accumulation point. We take this to be e_{m+1} and consider again all the directions of Ξ .

Repeating the process there is a segment whose $(m+1)$ -shadow is not (n, p) -lacunary separated from the accumulation point of accumulation points. This gives rise to another sequence of accumulation points, and an accumulation point of them, separated from the original accumulation point of accumulation points. Dividing them either yields a segment whose $(m+1)$ -shadow is not (n, p) -lacunary and is separated from them, and we continue the process with this, or there is a segment whose $(m+1)$ -shadow is not (n, p) -lacunary and which contains one of the accumulation points of accumulation points, which we take to be e_{m+1} , and continue the process with this. Eventually this yields a full sequence of accumulation points of accumulation points, which have an accumulation point by compactness. Continuing the process, we see that Ξ contains accumulation points of arbitrarily large order which contradicts Lemma 2.2, and so we are done. \square

Note that if the constant 2 in the definition of a dominating segment is replaced by another which is strictly larger than one, the previous argument is unaffected. Thus, by Theorem B, we see that the finite order p -lacunary classes, defined with different dominating constants, are all the same.

3. CONCLUDING REMARKS

If Theorem A were more flexible, in the sense that the partitions were allowed to ‘accumulate’ away from the hyperplanes orthogonal to the basis vectors, then $p\text{-Lac}(n)$ and $\text{Lac2shad}(n)$ would be the same, as we would be

able to bound the operators associated to the latter class. However, Theorem A is remarkably sharp in the sense that the supremum in σ must be taken over the whole of Σ , and the partitions must accumulate at the hyperplanes perpendicular to the basis vectors. To see this, we construct sets of relatively well-behaved directions for which the associated maximal operators are unbounded. As in the previous section, the directions need only be badly spaced after projecting onto a two-dimensional subspace, and so, in contrast with the two-dimensional case, it is not enough to constrain the angles between the directions if they are to give rise to a bounded maximal operator in higher dimensions.

First, we enumerate $\mathbb{Q} \cap [\frac{1}{2}, \frac{2}{3}] = \{q_\ell\}_{\ell \geq 1}$ and consider

$$\Omega = \left\{ \omega \in \mathbb{S}^{n-1} \cap \mathbb{R}_+^n : \omega_1 = q_\ell \omega_2, \omega_j = 2^{-j\ell}, 1 < j < n, \ell \geq 1 \right\}.$$

Then the angles between the directions form a lacunary sequence converging to zero with lacunary constant $1/2$. Taking $\theta_{\sigma,i} = 2^{-i}$, the segments $\Omega_{\sigma,i}$ consist of at most one direction for all $i \in \mathbb{Z}^*$ and $\sigma \in \Sigma \setminus \{(1,2)\}$. In spite of this, M_Ω is unbounded. Indeed, consider the set of rectangles \mathcal{R} in $\Pi = \text{span}(e_1, e_2)$ with longest side parallel to the 2-shadow on Π of some $\omega \in \Omega$. Then the construction of Besicovitch provides finite subsets $\mathcal{R}_N \subset \mathcal{R}$, for all $N \geq 1$, that satisfy (6). Considering χ_{E_N} , defined as in the proof of Lemma 2.1, we find M_Ω unbounded as before.

Secondly, we let e'_2 and e'_n be orthogonal unit vectors in $\text{span}(e_2, e_n)$, close to e_2 and e_n , with e'_n in the first quadrant determined by e_2 and e_n . We construct a set of directions, accumulating rapidly at e'_n , for which the angles between the orthogonal projections onto $\text{span}(e_1, e'_2)$ are badly spaced. Indeed, we take $\Omega = \{\omega_\ell\}_{\ell \geq 1}$ so that $\omega_\ell \cdot e'_2 = q_\ell \omega_\ell \cdot e_1$. This does not yet completely determine ω_ℓ . Supposing that we have chosen $\omega_{\ell-1}$ we can choose the direction ω_ℓ sufficiently close to e'_n so that the angle between $\omega_{\ell-1}$ and e'_n is at least double that between ω_ℓ and e'_n . We can also choose the directions so that

$$\frac{\omega_{\ell-1} \cdot e'_n}{\omega_{\ell-1} \cdot e'_2} \leq \frac{1}{2} \frac{\omega_\ell \cdot e'_n}{\omega_\ell \cdot e'_2}, \quad \text{and} \quad \frac{\omega_{\ell-1} \cdot e_k}{\omega_{\ell-1} \cdot e_j} \leq \frac{1}{2} \frac{\omega_\ell \cdot e_k}{\omega_\ell \cdot e_j}$$

for all $(j, k) \in \Sigma(n) \setminus \{(2, n)\}$. Taking $\theta_{\sigma,i} = 2^{-i}$, the segments $\Omega_{\sigma,i}$, defined with respect to the orthonormal basis (e_1, \dots, e_n) , consist of at most one direction for all $i \in \mathbb{Z}^*$ and $\sigma \in \Sigma(n) \setminus \{(2, n)\}$. On the other hand, if we define the final segments by

$$\Omega_{(2,n),i} = \left\{ \omega \in \Omega : 2^{-(i+1)} < \left| \frac{\omega \cdot e'_n}{\omega \cdot e'_2} \right| \leq 2^{-i} \right\}, \quad i \in \mathbb{Z},$$

accumulating at $\{e'_2\}^\perp \cup \{e'_n\}^\perp$, then they also consist of at most one direction for all $i \in \mathbb{Z}$. In spite of this, M_Ω is unbounded as before. Indeed, consider the set of rectangles \mathcal{R} in $\Pi = \text{span}(e_1, e'_2)$ with longest side parallel to the 2-shadow on Π of some ω_ℓ . Then there are finite subsets $\mathcal{R}_N \subset \mathcal{R}$, for all $N \geq 1$, that satisfy (6). Considering χ_{E_N} , defined as in the proof of

Lemma 2.1, but with respect to the basis $(e_1, e'_2, e_3, \dots, e_{n-1}, e'_n)$, we again find M_Ω unbounded on $L^p(\mathbb{R}^n)$ for finite p .

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